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A Proof of the Borsuk Antipodal Theorem for Fredholm Maps

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With only the means of elementary analysis and the Smale–Sard lemma, a direct and self-contained proof of the Borsuk Antipodal Theorem for Fredholm maps is given. The discussion extends the arguments of J. C. Alexander and J. A. Yorke used in their analytical proof of the Antipodal Theorem in $\mathbb{R}^n$. (*Trans. Amer. Math. Soc.* **242** (1978), 271–284.)

1. PRELIMINARIES

We begin by recalling the terminology and some facts which will be used in our discussion.

Notation. In what follows E and F denote real Banach spaces; $E \oplus F$ stands for the *direct product* of E and F: every $x \in E \oplus F$ can be uniquely written as $x \in x_1 + x_2$ with $x_1 \in E$, $x_2 \in F$. The map $P_E = P: E \oplus F \to E$ given by $x_1 + x_2 \longmapsto x_1$ is the *projection on E along F*.

By $L(E, F)$ we denote the Banach space of all continuous linear maps $A: E \to F$ with $\|A\| = \sup\{\|Ax\|; \|x\| \leqslant 1\}$. We let $K(E, F)$ be the subspace of $L(E, F)$ consisting of all completely continuous (=compact) operators. We let $L(E) = L(E, E)$, $K(E) = K(E, E)$ and we denote by $GL(E)$ the group of invertible operators in $L(E)$.

Fredholm operators. An operator $A \in L(E, F)$ is *Fredholm* if $\operatorname{Im} A \subset F$ is closed and both $\operatorname{Ker} A$ and $\operatorname{Coker} A$ are finite dimensional. For such an A the *index* $\operatorname{ind} A = \dim \operatorname{Ker} A - \dim \operatorname{Coker} A$. We let $\Phi_k(E, F)$ denote the set of all Fredholm operators from E to F of index k. In this paper unless otherwise stated Fredholm operators are always assumed to be of index 0.

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We now collect some elementary facts on Fredholm operators; these are stated only in such generality as will be used in the sequel.

(1.1). *If $A \in \Phi_0(E, F)$ and $K \in K(E, F)$ then $A + K \in \Phi_0(E, F)$.*

(1.2). *If $A \in \Phi_0(E, F)$ then there are closed subspaces $E_0 \subset E$, $F_0 \subset F$ and $B \in \Phi_0(F, E)$ such that*

(i) $E = E_0 \oplus \operatorname{Ker} A$,

(ii) $F = \operatorname{Im} A \oplus F_0$,

(iii) $\operatorname{Im} B = E_0$, $\operatorname{Ker} B = F_0 \approx \operatorname{Ker} A$,

(iv) $A \cdot B = \mathrm{Id}_{\operatorname{Im} A}$, $B \cdot A = \mathrm{Id}_{E_0}$.

For a proof of (1.1), (1.2) see, for example, Schechter [4].

(1.3). *If $A \in \Phi_0(E)$ then for each $\varepsilon > 0$ there is $K \in K(E)$ with $\|K\| < \varepsilon$ such that $A + K \in GL(E)$.*

Proof. Let $E = F$ and E_0, F_0 be as in (1.2). Let P be the projection onto $\operatorname{Ker} A$ along E_0 and let $\varphi: \operatorname{Ker} A \to F_0$ be a linear isomorphism. Set $K_0 = \varphi \cdot P$. Then for any $\lambda > 0$, $A + \lambda K_0 \in GL(E)$; choose λ so small that $\lambda \cdot \|K_0\| < \varepsilon$ and let $K = \lambda K_0$. Then K has the desired property.

(1.4). *Let $A: E \oplus F \to E$ be a linear surjective operator and let $P: \operatorname{Ker} A \to F$ be the restriction to* $\operatorname{Ker} A$ *of the projection onto F along E. Then*

(i) *$A|_E : E \approx E$ if and only if $P: \operatorname{Ker} A \approx F$,*

(ii) *if $A|_E \in \Phi_0(E)$ then $P \in \Phi_0(\operatorname{Ker} A, F)$.*

Proof. (i) If $A_0 = A|_E$ is an isomorphism, we define $B: F \to \operatorname{Ker} A$ by $y \to (-A_0^{-1}Ay, y)$; one verifies easily $P \cdot B = \mathrm{Id}_F$ and $B \cdot P = \mathrm{Id}_{\operatorname{Ker} A}$. The converse implication is evident.

(ii) Since A is surjective there is a decomposition $F = F_0 \oplus F_1$ such that $A(E \oplus F_1) = \operatorname{Im} A$ and $\dim F_0 = \dim \operatorname{Ker} A_0 < \infty$. Put $E = E_0 \oplus \operatorname{Ker} A_0$, $A_1 = A|_{E_0 \oplus F_1}: E_0 \oplus F_1 \to \operatorname{Im} A$. Since A is surjective and $A_1|_{E_0}$ is an isomorphism, we get by (i) that $\operatorname{Ker} A_1 \simeq F_1$. The conclusion now follows by observing that $\operatorname{Ker} A = \operatorname{Ker} A_0 \oplus \operatorname{Ker} A_1$.

Fredholm maps. Let U be a connected subset of E and $f: U \to F$ a C^1-map. A point $x \in U$ is called a *regular point for f* if $Df(x) \in L(E, F)$ is surjective and *critical* if it is not regular. The images of critical points are called *critical values.* A point $y \in F$ is said to be a *regular value* for f if it is not a critical value.

A C^1-map $f: U \to F$ is *Fredholm* provided every differential $Df(x)$ is a Fredholm operator, i.e., $Df(x) \in \Phi(E, F)$ for each $x \in U$. Then $\operatorname{ind} Df(x)$ is constant on U and the common value of $\operatorname{ind} Df(x)$ (denoted by $\operatorname{ind} f$) is called the *index of* f. The family of all Fredholm C^n-maps of index n will be denoted by $\Phi_n C^n$.

All the above definitions generalize to the context of Banach manifolds (cf. Lang [3]). A C^1-map $f: M \to N$ between connected Banach manifolds is Fredholm provided each derivative $Df(x): T_x M \to T_{f(x)} N$ is a Fredholm operator ($T_x M =$ the tangent space to M at $x \in M$). The index of f is defined to be the index of $Df(x)$ for some x; since M is connected this definition does not depend on x. All other definitions (regular points, regular values, etc.,) extend to this more general case without changes.

Proper maps. A map $f: X \to Y$ is *proper* provided for each compact $K \subset Y$, $f^{-1}(K)$ is compact in X; f is σ-proper if $X = \bigcup_{i \geqslant 1} X_i$ with X_i closed and $f|_{X_i}$ a proper map. A map $f: X \to Y$ is *locally proper* provided for each $x \in X$ there is a closed neighbourhood N_x of X such that $f|_{N_x}: N_x \to Y$ is proper.

(1.5). *Some frequently used properties of proper maps are*:

(i) *every proper map is closed* (*i.e., sends closed sets into closed sets*),

(ii) *every Fredholm map is locally proper,*

(iii) *every perturbation of a proper map by a compact map is also proper.*

The topological degree for admissible Fredholm maps. A map $f: (\bar{U}, \partial U) \to (E, E\setminus\{0\})$ is called *admissible* if

(i) f is proper,

(ii) $f|_U: U \to E$ is a Fredholm C^2-map of index 0.

Two admissible maps f, $g: (\bar{U}, \partial U) \to (E, E\setminus\{0\})$ are said to be *homotopic* (written $f \sim g$) provided there exists a map $h: (\bar{U}, \partial U) \times [0, 1] \to (E, E\setminus\{0\})$ such that

(i) h is proper,

(ii) $h|_{U \times [0,1]}: U \times [0, 1] \to E$ is of class C^2,

(iii) $Dh(x, t) \in \Phi_1(E \oplus R, E)$ for each $(x, t) \in U \times [0, 1]$,

(iv) $h(x, 0) = f(x)$, $h(x, 1) = g(x)$ for each $x \in \bar{U}$.

h is called an *admissible homotopy* connecting f and g.

We recall now the definition of the mod 2 topological degree for admissible maps. The main tool is the following theorem which represents a consequence of Smale's generalization of Sard's lemma (cf. Smale [5]):

(1.6) THEOREM. *Let $f: (\bar{U}, \partial U) \to (E, E\backslash\{0\})$ be an admissible map. Then the subset of $E\backslash f(\partial U)$ consisting of regular values of $f|_U$ is open and dense in $E\backslash f(\partial U)$.*

Let $f: (\bar{U}, \partial U) \to (E, E\backslash\{0\})$ be an admissible map and $y \in E$ be a regular value for $f|_U$, such that $\|y\| < \inf\{\|f(x)\|; x \in \partial U\}$. Then it follows that $f^{-1}(y)$ is either empty or consists of a finite number of points $x_1, x_2, \ldots, x_k \in U$. The *degree of f* with respect to y (written $\deg(f, y)$) is defined to be zero if $f^{-1}(y) = \varnothing$ and $\deg(f, y) = k \pmod 2$.

The degree $\deg(f, y)$ is independent of the choice of the regular value y; we denote this invariant by $\deg(f)$. The main properties of the degree are:

(i) $\deg f \neq 0 \Rightarrow f(x) = 0$ for some $x \in U$,

(ii) $f \sim g \Rightarrow \deg f = \deg g$.

2. ADMISSIBLE HOMOTOPIES

In what follows, by U we denote a bounded domain in E containing 0, and by $f: (\bar{U}, \partial U) \to (E, E\backslash\{0\})$ an arbitrary but fixed admissible map. Given a compact operator $A \in K(E)$ we let $f_A = f + A$; clearly, every such f_A is a proper map.

The aim of this section is the following:

(2.1) THEOREM. *Let f be an admissible map. There exists a linear compact operator $A \in K(E)$ such that*

(i) *the map $f_A = f + A$ is admissible*;

(ii) $f \sim f_A$;

(iii) *0 is a regular value of f_A*.

The proof of (2.1) is based on two lemmas.

(2.2) LEMMA. *Let f be an admissible map. There exists an $\varepsilon > 0$ satisfying*

(i) *for each $A \in K(E)$ with $\|A\| < \varepsilon$, the map f_A is admissible and $f \sim f_A$.*

(ii) *for some $A \in K(E)$ with $\|A\| < \varepsilon$, the derivative $Df_A(0) \in GL(E)$.*

Proof. (i) Since f is proper, $f(\partial U)$ is closed in E and hence $\operatorname{dist}(f(\partial U), 0) = d > 0$. Let $M > 0$ be such that $\|x\| \leqslant M$ for $x \in \partial U$ and put $\varepsilon = d/M$. Let $\|A\| < \varepsilon$, $A \in K(E)$; we have for $x \in \partial U$

$$\|f_A(x)\| = \|f(x) + Ax\| \geqslant \|fx + Ax\| > \|f(x)\| - d \geqslant 0.$$

Thus $f_A:(\bar{U}, \partial U) \to (E, E\backslash\{0\})$. Since on U we evidently have $f_A \in \Phi_0 C^2$, the map f_A is admissible. To prove $f \sim f_A$, we observe that

$$h(x, t) = f(x) + tA(x)$$

defines an admissible homotopy joining f and f_A.

(ii) Follows at once from (i) and (1.3).

(2.3) LEMMA. *Let $g:(\bar{U}, \partial U) \to (E, E\backslash\{0\})$ be an admissible map satisfying $Dg(0) \in GL(E)$. Let V be an open neighbourhood of* 0 *in $K(E)$ such that for all $A \in V$ the map $g_A = g + A$ is admissible and $Dg_A(0) \in GL(E)$. Define $G:U \times V \to E$ by $G(x, A) = G(x) + A(x)$. Then*

(i) *G is differentiable and for $(h, H) \in E \times K(E)$ we have $DG(x, A) * (h, H) = Dg(x) * h + A(h) + H(x)$.*

(ii) *every $(x, A) \in U \times V$ is a regular point for G,*

(iii) *$M = G^{-1}(0)$ is a submanifold of $U \times V$ and the projection $\pi:M \to V$ is a Fredholm map of index* 0.

(iv) *for each open $V_0 \subset V$ an operator $A \in V_0$ is a regular value of the restriction $\pi:\pi^{-1}(V_0) \to V_0$ if and only if* 0 *is a regular value of g_A.*

Proof. (i) This follows at once from the definition of G.

(ii) We have to prove that for every $(x, A) \in U \times V$, $DG(x, A)$ is surjective. If $x = 0$ then $DG_E(0, A) = Dg_A(0)$, which is surjective by the assumption. If $x \neq 0$ then $DG(x, A) * (hH) = H(x)$. Since for any $v \in E$ one can find $H \in K(E)$ such that $H(x) = v$, $DG(x, A)|_{K(E)}$ is surjective.

(iii) Since 0 is a regular value of G, M is submanifold of $U \times V$ and $T_{(x,A)}M = \operatorname{Ker} DG(x, A)$. Applying (1.4) to $DG(x, A):E \times K(E) \to E$ we obtain the desired conclusion.

(iv) Clearly A is a regular value of $\pi:\pi^{-1}(V_0) \to V_0$ if and only if $\operatorname{Ker} DG(x, A) \cap E = \{0\}$ for all $x \in g_A^{-1}(0)$. By (i) of (1.4) this is true if and only if 0 is a regular value of g_A.

Proof of Theorem 2.1. We remark first that in the proof of (2.1) we shall again apply the Smale–Sard lemma. More precisely we shall use it in the following (suitable for our purpose) local form:

> If M, N are Banach manifolds and $\pi:M \to N$ is a C^1-Fredholm map of index 0, then for each $x \in M$ there is a neighbourhood $U_x \subset M$ of x such that the set of regular values of the restriction $\pi|_{U_x}:U_x \to N$ is a Baire subset of N. $(*)$

We begin the proof by applying Lemma (2.3): we replace f by g satisfying the assumptions of (2.3). By Lemma (2.3) it will be enough to show that for some open $V_0 \subset V$ there exists an operator $A \in V_0$ such that A is a regular value of the restriction $\pi_0 = \pi|_{\pi^{-1}(V_0)} : \pi^{-1}(V_0) \to V_0$. To find such a V_0, we shall use the local form of the Sard lemma applied to the projection $\pi: M \to V$.

Putting (using the notation of (2.3))

$$C = M \cap [U \times \{0\}] = g_A^{-1}(0) \times \{0\}$$

and observing that $C \subset M$ is compact, we find, using $(*)$, a finite family $\{W_i\}_{i=1}^r$ of open sets $W_i \subset M$ satisfying $C \subset \bigcup_{i=1} W_i$ and such that the set of regular values of $\pi|_{W_i}$ is a Baire subset of $V \subset K(E)$. Putting $W = \bigcup_{i=1}^r W_i$, we now claim that there is an open neighbourhood $V_0 \subset V$ of the origin $0 \in K(E)$ such that $\pi^{-1}(V_0) \subset W$. For, suppose this is not true. Then we could find two sequences $\{A_n\}$, $A_n \in V \subset K(E)$ and $\{x_n\}$, $x_n \in U$ such that (i) $A_n \to 0$, (ii) $(x_n, A_n) \in M \backslash W$. Clearly, because U is bounded, we have $A_n(x_n) \to 0$. Thus, since $(x_n, A_n) \in M$ we have $g(x_n) = -A(x_n)$. Since g is proper it follows that there is a subsequence $x_{n_k} \to x_0$. Thus, $g(x_0) = 0$ and since $x_0 \notin \partial U$ we get $(x_0, 0) \in C$.

But, because of (ii), this a contradiction. Thus for some open $V_0 \subset V$, we have $\pi^{-1}(V_0) \subset W$, and consequently the restriction map $\pi : \pi^{-1}(V_0) \to V_0$ has at least one regular value $A \in V_0$.

The proof of Theorem (2.1) is complete.

3. The Antipodal Theorem

Now we formulate and prove a theorem, which extends to "Fredholm maps the Antipodal Theorem of Borsuk."

(3.1) Theorem. *Let U be a bounded domain in E, containing* 0 *and symmetric (i.e., $-U = U$). Let $f : (\bar{U}, \partial U) \to (E, E\backslash\{0\})$ be an admissible map such that $f(-x) = -f(x)$ for all $x \in \bar{U}$. Then* $\deg(f) \equiv 1 \pmod 2$.

Proof. By Theorem (2.1) there exists an operator $A \in K(E)$ such that $f_A = f + A$ is an admissible map, $f_A \sim f$, and such that 0 is a regular value of f_A. Since $f_A(-x) = -f_A(x)$ for all $x \in \bar{U}$ we have

$$f_A^{-1}(0) = \{0, x_1, -x_1, \ldots, x_k, -x_k\}.$$

Thus $\deg f = \deg f_A \equiv 1 \pmod 2$. The proof is complete.

In conclusion we remark that Theorem (3.1) was announced by Elworthy and Tromba [2].

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